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Generation of gravitational waves. Linear momentum flux to higher orders

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Abstract. A new explicit expression of linear momentum flux for an isolated radiating system of N bodies (or for a perfect fluid) in the linear approximation within general relativity is obtained up to higher orders.

1. Introduction

In two earlier papers (Dionysiou 1974, 1975) the linear momentum flux up to octupole order is given by

$$P^{\alpha} = \frac{1}{315}\left(22\dddot{D}^{\beta\gamma}\frac{\partial^4}{\partial t^4}(D^{\beta\gamma}x^{\alpha}) - 12\dddot{D}^{\beta\gamma}\frac{\partial^4}{\partial t^4}(D^{\alpha\beta}x^{\gamma}) - 12\dddot{D}^{\alpha\beta}\frac{\partial^4}{\partial t^4}(D^{\beta\gamma}x^{\gamma})\right), \tag{1}$$

where P^{α} is the total outflux per unit time of the αth (Cartesian) component of linear momentum. Greek indices run from 1 to 3, Latin indices from 0 to 3 and we set $G = c = 1$. The dots mean derivatives with respect to time. Also

$$D^{\alpha\beta} = I^{\alpha\beta} - \tfrac{1}{3}\delta_{\alpha\beta}I^{\gamma\gamma} \tag{2}$$

is the quadrupole moment tensor and

$$I^{\alpha\beta} = \sum_{m} m x_{\alpha} x_{\beta} \tag{3}$$

is the mass tensor of the system.

Particularly we note that the previous results (Bonnor and Rotenberg 1961, Papapetrou 1962, Peres 1962) are similar but less general than equation (1) (Bonnor and Rotenberg 1961, Dionysiou 1975).

2. Linear momentum flux

In this paper we generalize equation (1). In a weak-field approximation to general relativity, far outside the source, the field equations can be written in the form

$$n^{lm}\gamma^{ij}_{,lm} = 16\pi\Theta^{ij}_{\mathrm{N}}, \tag{4}$$

where $\Theta^{ij}_{\mathrm{N}} = (T^{ij} + t^{ij})_{\mathrm{N}}$ are the dominant Newtonian terms of the Landau–Lifshitz complex $\Theta^{ij} = (-g)(T^{ij} + t^{ij})$. Here, we impose on γ^{ij} the gauge condition $\gamma^{ij}{}_{,j} = 0$, where

$$\sqrt{-g}g^{ij} = \eta^{ij} + \gamma^{ij}, \qquad g^{ij} = \eta^{ij} + h^{ij}, \qquad |h^{ij}| \ll 1$$

and η^{ij} the Minkowskian metric. Ordinary differentiation is denoted by a comma.

The solution of equation (4), which satisfies the outgoing radiation condition, is (Misner *et al* 1973, p 996):

$$\gamma^{ij}(\boldsymbol{x}, t) = 4 \int_{\text{all space}} \frac{\Theta_{\mathrm{N}}^{ij}(\boldsymbol{x}', t - |\boldsymbol{x} - \boldsymbol{x}'|)}{|\boldsymbol{x} - \boldsymbol{x}'|} \, \mathrm{d}\boldsymbol{x}' \tag{5a}$$

or

$$\gamma^{ij}(\boldsymbol{x}, t) = \frac{4}{r} \int_{\text{all space}} \Theta_{\mathrm{N}}^{ij}(\boldsymbol{x}', t - r + (\boldsymbol{n} \,.\, \boldsymbol{x}')) \, \mathrm{d}\boldsymbol{x}' + \mathrm{O}(r^{-2}), \tag{5b}$$

where $|\boldsymbol{x} - \boldsymbol{x}'| = r - \boldsymbol{n} \,.\, \boldsymbol{x}' + \mathrm{O}(r^{-1})$, $r = |\boldsymbol{x}|$ and $\boldsymbol{n} = \boldsymbol{x}/r$ is the unit vector in the direction of propagation.

Equation (5*b*) can be written as an expansion

$$\gamma^{ij}(\boldsymbol{x}, t) = \frac{4}{r} \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \int_{\text{all space}} \left(\frac{\partial^{\nu}}{\partial t^{\nu}} \Theta_{\mathrm{N}}^{ij}(\boldsymbol{x}', t - r) \right) (\boldsymbol{n} \,.\, \boldsymbol{x}')^{\nu} \, \mathrm{d}\boldsymbol{x}' + \mathrm{O}(r^{-2}) \tag{5c}$$

and if the motion of the particles is sufficiently slow, equation (5*c*) may be replaced by an expansion

$$\gamma^{ij}(\boldsymbol{x}, t) = \frac{4}{r} \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \frac{\partial^{\nu}}{\partial t^{\nu}} \int_{\text{all space}} \Theta_{\mathrm{N}}^{ij}(\boldsymbol{x}', t - r)(\boldsymbol{n} \,.\, \boldsymbol{x}')^{\nu} \, \mathrm{d}\boldsymbol{x}' + \mathrm{O}(r^{-2}). \tag{5d}$$

One can put the conservation laws with the help of the special form

$$\Theta_{\mathrm{N},j}^{ij} = 0. \tag{6}$$

Applying equation (6) one obtains the identities:

$$\frac{\partial^2 \Theta_{\mathrm{N}}^{00}}{\partial t^2} \equiv \frac{\partial^2 \Theta_{\mathrm{N}}^{\alpha\beta}}{\partial x'_{\alpha} \partial x'_{\beta}} \tag{7a}$$

$$\frac{\partial^2}{\partial t^2}(\Theta_{\mathrm{N}}^{00} x'_{\alpha} x'_{\beta}) \equiv \frac{\partial^2}{\partial x'_{\gamma} \partial x'_{\delta}}(\Theta_{\mathrm{N}}^{\gamma\delta} x'_{\alpha} x'_{\beta}) - 2 \frac{\partial}{\partial x'_{\gamma}}(\Theta_{\mathrm{N}}^{\gamma\alpha} x'_{\beta} + \Theta_{\mathrm{N}}^{\gamma\beta} x'_{\alpha}) + 2\Theta_{\mathrm{N}}^{\alpha\beta}. \tag{7b}$$

We suppose Θ^{ij} spatially confined as it is required by equation (5*a*) (Misner *et al* 1973, pp 989–1001); then the first and second terms on the right-hand side of equation (7*b*) are seen to be zero, when we integrate over all space by an application of Gauss' theorem.

Hence from equations (7*a*) and (7*b*) follow:

$$\int \Theta_{\mathrm{N}}^{\alpha\beta} \, \mathrm{d}\boldsymbol{x}' = \frac{1}{2} \frac{\partial^2}{\partial t^2} \int \Theta_{\mathrm{N}}^{00} x'_{\alpha} x'_{\beta} \, \mathrm{d}\boldsymbol{x}' \tag{8}$$

$$\begin{aligned} \int \Theta_{\mathrm{N}}^{\alpha\beta} x'_{\kappa} \, \mathrm{d}\boldsymbol{x}' = {} & \frac{1}{6} \frac{\partial^2}{\partial t^2} \int \Theta_{\mathrm{N}}^{00} x'_{\alpha} x'_{\beta} x'_{\kappa} \, \mathrm{d}\boldsymbol{x}' \\ & - \frac{1}{3} \frac{\partial}{\partial t} \int [(\Theta_{\mathrm{N}}^{0\kappa} x'_{\beta} - \Theta_{\mathrm{N}}^{0\beta} x'_{\kappa}) x'_{\alpha} + (\Theta_{\mathrm{N}}^{0\kappa} x'_{\alpha} - \Theta_{\mathrm{N}}^{0\alpha} x'_{\kappa}) x'_{\beta}] \, \mathrm{d}\boldsymbol{x}', \end{aligned} \tag{9}$$

where Θ_{N}^{ij} under the integrals mean $\Theta_{\mathrm{N}}^{ij}(\boldsymbol{x}', t - r)$.

Using a transverse-traceless (TT) gauge condition i.e. (Misner *et al* 1973, pp 946–9)

$$h^{i0}=0, \qquad h^{\alpha\beta}{}_{,\beta}=0, \qquad h^{\alpha\alpha}=0$$

then the gravitational radiation is completely described by the TT part of the metric perturbation h^{ij}. Hence, since

$$h^{ij}=\gamma^{ij}-\tfrac{1}{2}\eta^{ij}\gamma+\text{non-linear expressions}$$

we have $h^{ij}_{\mathrm{TT}}=\gamma^{ij}_{\mathrm{TT}}$.

Since $\gamma^{\alpha\beta}_{\mathrm{TT}}$ is the gauge invariant TT part of γ^{ij}, then using equations (5*d*), (8) and (9) we have

$$\begin{aligned}\gamma^{\alpha\beta}_{\mathrm{TT}}(\boldsymbol{x},t)=&\frac{4}{r}\frac{1}{2}\frac{\partial^2}{\partial t^2}\int \Theta^{00}_{\mathrm{N}}x'_\alpha x'_\beta\,\mathrm{d}\boldsymbol{x}'+\frac{4}{r}n_{\kappa_1}\biggl(\frac{1}{6}\frac{\partial^3}{\partial t^3}\int \Theta^{00}_{\mathrm{N}}x'_\alpha x'_\beta x'_{\kappa_1}\,\mathrm{d}\boldsymbol{x}'-\frac{1}{3}\frac{\partial^2}{\partial t^2}\\ &\times\int[(\Theta^{0\kappa_1}_{\mathrm{N}}x'_\beta-\Theta^{0\beta}_{\mathrm{N}}x'_{\kappa_1})x'_\alpha+(\Theta^{0\kappa_1}_{\mathrm{N}}x'_\alpha-\Theta^{0\alpha}_{\mathrm{N}}x'_{\kappa_1})x'_\beta]\,\mathrm{d}\boldsymbol{x}'\biggr)+\frac{4}{r}\frac{1}{2!}n_{\kappa_1}n_{\kappa_2}\frac{\partial^2}{\partial t^2}\\ &\times\int\Theta^{\alpha\beta}_{\mathrm{N}}x'_{\kappa_1}x'_{\kappa_2}\,\mathrm{d}\boldsymbol{x}'+\ldots+\frac{4}{r}\frac{1}{\nu!}n_{\kappa_1}n_{\kappa_2}n_{\kappa_3}\ldots n_{\kappa_\nu}\frac{\partial^\nu}{\partial t^\nu}\\ &\times\int\Theta^{\alpha\beta}_{\mathrm{N}}x'_{\kappa_1}x'_{\kappa_2}\ldots x'_{\kappa_\nu}\,\mathrm{d}\boldsymbol{x}'+\mathrm{O}(r^{-2}),\end{aligned}\tag{10}$$

where

$$\Theta^{00}_{\mathrm{N}}=\sum_{m'}m'\delta(\boldsymbol{x}-\boldsymbol{x}'), \qquad \Theta^{0\alpha}_{\mathrm{N}}=\sum_{m'}m'u'_\alpha\delta(\boldsymbol{x}-\boldsymbol{x}')$$

and

$$\Theta^{\alpha\beta}_{\mathrm{N}}=\sum_{m'}m'u'_\alpha u'_\beta\delta(\boldsymbol{x}-\boldsymbol{x}')+t^{\alpha\beta}_{\mathrm{N}}$$

(Dionysiou 1974), or

$$\gamma^{\alpha\beta}_{\mathrm{TT}}(\boldsymbol{x},t)=\frac{2}{r}\sum_{\nu=0}^{\infty}n_{\kappa_1}n_{\kappa_2}\ldots n_{\kappa_\nu}\frac{\partial^2}{\partial t^2}I^{\alpha\beta\kappa_1\kappa_2\ldots\kappa_\nu}_{\mathrm{TT}}(t-r)+\mathrm{O}(r^{-2}),\tag{11}$$

where

$$I^{\alpha\beta}=\sum_m m x_\alpha x_\beta,\tag{12}$$

$$I^{\alpha\beta\kappa_1}=\sum_m m u_\alpha x_\beta x_{\kappa_1}+\sum_m m x_\alpha u_\beta x_{\kappa_1}-\sum_m m x_\alpha x_\beta u_{\kappa_1}\tag{13}$$

and

$$I^{\alpha\beta\kappa_1\kappa_2\ldots\kappa_\nu}=\frac{2}{\nu!}\frac{\mathrm{d}^{\nu-2}}{\mathrm{d}t^{\nu-2}}\int\Theta^{\alpha\beta}_{\mathrm{N}}x'_{\kappa_1}x'_{\kappa_2}x'_{\kappa_3}\ldots x'_{\kappa_\nu}\,\mathrm{d}\boldsymbol{x}', \qquad \nu\geqslant 2.\tag{14}$$

With the aid of equations (12), (13) and (14) we can define

$$D^{\alpha\beta\kappa_1\kappa_2\ldots\kappa_\nu}=I^{\alpha\beta\kappa_1\kappa_2\ldots\kappa_\nu}-\tfrac{1}{3}\delta_{\alpha\beta}I^{\gamma\gamma\kappa_1\kappa_2\ldots\kappa_\nu},\tag{15}$$

which readily gives the following properties:

$$D^{\alpha\alpha\kappa_1\kappa_2\ldots\kappa_\nu}=0\tag{16}$$

$$D^{\alpha\beta\kappa_1\kappa_2\ldots\kappa_\nu}_{\mathrm{TT}}=I^{\alpha\beta\kappa_1\kappa_2\ldots\kappa_\nu}_{\mathrm{TT}}\tag{17}$$

whose 'TT' projection is on the α,β indices.

Substituting equation (17) into equation (11), one gets that

$$\gamma_{\mathrm{TT}}^{\alpha\beta}(\boldsymbol{x},t)=\frac{2}{r}\sum_{\nu=0}^{\infty} n_{\kappa_1\kappa_2}\ldots n_{\kappa_\nu}\ddot{D}_{\mathrm{TT}}^{\alpha\beta\kappa_1\kappa_2\ldots\kappa_\nu}+\mathrm{O}(r^{-2}), \tag{18}$$

where $D_{\mathrm{TT}}^{\alpha\beta}=P^{\alpha\gamma}D^{\gamma\delta}P^{\beta\delta}-\frac{1}{2}P^{\alpha\beta}P^{\gamma\delta}D^{\gamma\delta}$, $P^{\alpha\beta}=\delta^{\alpha\beta}-n^\alpha n^\beta$, $n^\alpha=x^\alpha/r$ (Misner *et al* 1973, pp 989–1001).

For the outflux of linear momentum one may concentrate on the flux of the radial component n_α (Bekenstein 1973)

$$P^\alpha=\int_{\mathrm{sphere}}\Theta_{\mathrm{GW}}^{00}r^2 n_\alpha\,\mathrm{d}\Omega, \tag{19}$$

where $\Theta_{\mathrm{GW}}^{00}$ is the effective stress-energy tensor for the outgoing waves, i.e.

$$\Theta_{\mathrm{GW}}^{00}=\frac{1}{32\pi}\langle\gamma_{\mathrm{TT},0}^{\alpha\beta}\gamma_{\mathrm{TT},0}^{\alpha\beta}\rangle, \tag{20}$$

where '$\langle\ \rangle$' denotes an average over several wavelengths (Misner *et al* 1973, pp 989–1001).

Putting equation (18) into equation (20) it follows that (Epstein and Wagoner 1975)

$$\Theta_{\mathrm{GW}}^{00}=\frac{1}{8\pi r^2}\Big\langle\sum_{\nu,\rho=0}^{\infty} n_{\kappa_1}n_{\kappa_2}\ldots n_{\kappa_\nu}n_{\lambda_1}n_{\lambda_2}\ldots n_{\lambda_\rho}(\dddot{D}^{\alpha\beta\kappa_1\kappa_2\ldots\kappa_\nu}\dddot{D}^{\alpha\beta\lambda_1\lambda_2\ldots\lambda_\rho}-2n_\gamma n_\delta\dddot{D}^{\beta\gamma\kappa_1\kappa_2\ldots\kappa_\nu}$$
$$\times\dddot{D}^{\beta\delta\lambda_1\lambda_2\ldots\lambda_\rho}+\tfrac{1}{2}n_\alpha n_\beta n_\gamma n_\delta\dddot{D}^{\alpha\beta\kappa_1\kappa_2\ldots\kappa_\nu}\dddot{D}^{\gamma\delta\lambda_1\lambda_2\ldots\lambda_\nu})\Big\rangle. \tag{21}$$

The integral in equation (19) being taken over a large sphere of radius r, with n_α the three components of the outward normal and $\mathrm{d}\Omega$ the differential solid angle. Hence, substituting equation (21) into equation (19), we obtain that

$$P^\alpha=\frac{1}{105}(22\dddot{D}^{\beta\gamma}\dddot{D}^{\beta\gamma\alpha}-12\dddot{D}^{\beta\gamma}\dddot{D}^{\alpha\beta\gamma}-12\dddot{D}^{\alpha\beta}\dddot{D}^{\beta\gamma\gamma})+\text{terms with 7, 9, 11, \ldots indices}, \tag{22}$$

using the well known integrals

$$\int_{\mathrm{sphere}} n_{\kappa_1}n_{\kappa_2}\ldots n_{\kappa_\nu}\,\mathrm{d}\Omega=\frac{4\pi}{1\,.\,3\,.\,5\ldots(\nu+1)}\Delta_{\kappa_1\kappa_2\ldots\kappa_\nu}\qquad(\nu\ \text{even}) \tag{23}$$

where $\Delta_{\kappa_1\kappa_2\ldots\kappa_\nu}$ means all distinct permutations of $\delta_{\kappa_1\kappa_2}$, and

$$\int_{\mathrm{sphere}} n_{\kappa_1}n_{\kappa_2}\ldots n_{\kappa_\nu}\,\mathrm{d}\Omega=0\qquad(\nu\ \text{odd}). \tag{24}$$

Hence, we can find terms with 7, 9, 11, ... indices, i.e. the linear momentum flux up to higher orders.

3. Conclusions

It is obvious that equation (22) is the general equation of the linear momentum loss from an N-body isolated system, where $D^{\alpha\beta\kappa_1\kappa_2\ldots\kappa_\nu}$ is defined by equation (15) (Dionysiou 1974, Papapetrou 1962). At this point we note that the higher multipoles decrease rapidly in magnitude. Also, if the nature of the matter behaves like a perfect fluid, then one will be able to define:

$$I^{\alpha\beta}(t)=\int \rho(\boldsymbol{x}',t)x'_\alpha x'_\beta\,\mathrm{d}\boldsymbol{x}', \tag{25}$$

$$I^{\alpha\beta\kappa_1}(t)=\int [\rho(\boldsymbol{x}',t)u'_\alpha x'_\beta x'_{\kappa_1}+\rho(\boldsymbol{x}',t)x'_\alpha u'_\beta x_{\kappa_1}-\rho(\boldsymbol{x}',t)x'_\alpha x'_\beta u'_{\kappa_1}]\,\mathrm{d}\boldsymbol{x}' \tag{26}$$

and

$$I^{\alpha\beta\kappa_1\kappa_2\ldots\kappa_\nu}(t)=\frac{2}{\nu!}\frac{\mathrm{d}^{\nu-2}}{\mathrm{d}t^{\nu-2}}\int \Theta_{\mathrm{N}}^{\alpha\beta}(\boldsymbol{x}',t)x'_{\kappa_1}x'_{\kappa_2}\ldots x'_{\kappa_\nu}\,\mathrm{d}\boldsymbol{x}' \qquad \nu\geqslant 2, \tag{27}$$

instead of equations (12), (13) and (14) respectively.

Then it is easy to see that we can obtain the same result, i.e. equation (22). We mention, without going into any details, that parallel results to higher orders have been obtained by the author for energy and angular momentum loss (Dionysiou 1977a, b).

Finally, the integrals (5*a*), (5*b*), (5*c*) and (5*d*) exist, since we can make the assumption that Θ_{N}^{ij} is spatially confined. (In the radiation zone we have $\Theta^{ij}\sim t^{ij}\sim 1/r^2$. Such contributions are ignored in our calculations as a second-order effect (see Misner *et al* 1973, pp 989–1001, Ehlers *et al* 1976).

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